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# $K$-THEORY OF PROJECTIVE STIEFEL MANIFOLDS

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Abstract. Using the Hodgkin spectral sequence we calculate $K^*(X_{m,k})$, the complex $K$-theory of the projective Stiefel manifold $X_{m,k}$, for $mk$ even. For $mk$ odd, we are only able to calculate $K^0(X_{m,k})$, but this is sufficient to determine the order of the complexified Hopf bundle over $X_{m,k}$.

## 1. Introduction

In this paper we extend the calculation of $K^*(X_{m,k})$, the complex $K$-theory ring of projective Stiefel manifolds, begun by Antoniano, Gitler, Ucci and Zvengrowski in [2]. Prior to the appearance of [2], $K^*(X_{m,k})$ had been calculated for the real projective spaces $X_{m,1}$ ([1]) and projective orthogonal groups ([9]).

The work of Hodgkin [10], Roux [12] and, subsequently, Antoniano, Gitler, Ucci and Zvengrowski [2] emphasized the importance of the Hodgkin spectral sequence in calculating $K^*(G/H)$ where $G$ is a compact Lie group with $\pi_1(G)$ torsion free. Hodgkin laid the foundations for later work on the $K$-theory of homogeneous spaces and showed how to calculate $K^*(G/H)$ in various cases of interest. In [12] Roux dealt with the case of Stiefel manifolds viewed as $Spin(m)/Spin(m-k)$ (see also [7]). For the projective Stiefel manifolds $X_{4n,2s-1}$, the case treated in [2], the subgroup $H$ of $Spin(4n)$ is isomorphic to the group $Z/2 \times Spin(4n-2s+1)$.

In the general case considered in the present article $X_{m,k} = Spin(m)/H$, where $H$ contains $Spin(m-k)$ as a subgroup of index two but in general is not a product. To calculate $K^*(X_{m,k})$ the representation ring $RH$ is computed (§4, Theorem 2) together with the restriction homomorphism $RSpin(m) \to RH$. From this it can be deduced that the Hodgkin spectral sequence collapses (§7, Theorems 4 and 5) when $mk$ is even.

An important application is that the order of the complexified Hopf bundle associated to the double covering $V_{m,k} \to X_{m,k}$ may be calculated for all $m$ and $k$ (this order has been obtained in [2] only for $m = 4n$). Consequently, non-immersion results for projective Stiefel manifolds may be deduced as in [3], as well as alternative (somewhat shorter) proofs for the non-parallelizability results of [2].

## 2. Projective Stiefel manifolds as homogeneous spaces

The projective Stiefel manifold $X_{m,k}$ is the quotient of the Stiefel manifold $V_{m,k}$ (of orthonormal $k$-frames in $R^m$) by the involution which takes a $k$-frame $\{v_1, \cdots, v_k\}$ to its opposite $\{-v_1, \cdots, -v_k\}$. For $1 \leq k < m$, which we henceforth assume, the special orthogonal group $SO(m)$ acts transitively on $X_{m,k}$ with

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isotropy subgroup $K_{m,k}$ consisting of all matrices of $SO(m)$ of the form $\pm I \oplus A$, where $I = I_k$ is the $(k \times k)$ identity matrix and $A \in O(m-k)$.

Let $p : Spin(m) \to SO(m)$ be the 2-fold covering map. Then

$$X_{m,k} = Spin(m)/H_{m,k},$$

where $H_{m,k} = p^{-1}(K_{m,k})$ and $Spin(m-k)$ a subgroup of index 2 of $H_{m,k}$, is generated in the Clifford algebra by the last $(m-k)$ vectors of the canonical basis $\{e_1, \cdots, e_m\}$ of $R^m$. We have the commutative diagram

$$\begin{array}{ccccccc}
Spin(m-k-1) & \longrightarrow & H_{m,k+1} & \longrightarrow & Spin(m)d & \longrightarrow & X_{m,k+1} \\
\downarrow & & \downarrow & & & & \downarrow \\
Spin(m-k) & \longrightarrow & H_{m,k} & \longrightarrow & Spin(m) & \longrightarrow & X_{m,k}d \\
\downarrow p & & \downarrow p & & \downarrow p & & \\
SO(m-k) & \longrightarrow & K_{m,k} & \longrightarrow & SO(m) & \longrightarrow & X_{m,k}
\end{array}$$

The manifolds $X_{m,k}$ and $X_{m,k+1}$ are related as follows: if $Spin(m)/H_{m,k+1}$ is the corresponding expression for $X_{m,k+1}$, then $H_{m,k+1} \subset H_{m,k}$ and $Spin(m-k) \cap H_{m,k+1} = Spin(m-k-1)$.

The following description of $H_{m,k}$ will be useful. Let $\omega$ be an element of $H_{m,k}$ that is not contained in $Spin(m-k)$, so that $H_{m,k} = Spin(m-k) \sqcup \omega Spin(m-k)$. If at least one of $m$ or $k$ is even, $\omega$ can be chosen to lie in the center of $H_{m,k}$ . In the Clifford algebra, we choose $\omega = e_1 \cdots e_m$ if $m$ is even, and $\omega = e_1 \cdots e_k$ if $k$ is even and $m$ odd. Note that $\omega^2 = \pm 1$, depending on the values of $m$ and $k$ mod 4. In general

$$(e_1 \cdots e_r)^2 = \begin{cases} +1 \text{ if } r \equiv 0, 3 \pmod 4, \\ -1 \text{ if } r \equiv 1, 2 \pmod 4. \end{cases}$$

Let $\Omega$ be the subgroup of $H_{m,k}$ generated by $\omega$. If $\omega^2 = +1$ , then $\Omega = Z/2$ and $H_{m,k} \approx \Omega \times Spin(m-k)$. If both $m$ and $k$ are odd, $\omega$ cannot be chosen to lie in the center of $H_{m,k}$, and in this case we will choose $\omega = e_1 \cdots e_{k+1}$ . Then $\omega$ lies in the center of $H_{m,k+1}$, and $\omega^2 = \pm 1$ .

## 3. The complex representation ring $RSpin(m)$

Next we summarize the necessary facts on the complex representation ring of $Spin(m)$ from [4] and [11].

Let $m = 2n$ or $2n+1$, and write $T^n$ for the $n$-fold product $S^1 \times \cdots \times S^1$, where $S^1$ stands as usual for the circle group in the complex plane.

Let $z_i$ be the character given by projection of $T^n$ onto the $i$th factor. Then

$$RT^n = Z[z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}].$$

Let $\tau : T^n \to Spin(m)$ be the homomorphism given by

$$\tau(e^{i\theta_1}, \cdots, e^{i\theta_n}) = (\cos\theta_1 + e_1 e_2 \sin\theta_1) \ldots (\cos\theta_n + e_{2n-1} e_{2n} \sin\theta_n)$$

(product in the Clifford algebra). Then $\tau$ maps $T^n$ onto the maximal torus $T(n)$ of $Spin(m)$. Clearly $\tau(z_1, \cdots, z_n) = \pm 1$ if and only if each $z_i = \pm 1$, so $\tau$ covers its image $2^{n-1}$ times.

In $RSpin(m)[t]$ we define

$$\Lambda[t] = \sum_{i=0}^{m} t^i \Lambda^i = (1+t)^{\varepsilon} \prod_{i=1}^{n} (1+tz_i^2)(1+tz_i^{-2})$$

(where $\epsilon = 0$ for $m = 2n$ and $\epsilon = 1$ for $m = 2n+1$), and

$$\Delta_n[t] = \prod_{i=1}^{n} (z_i + tz_i^{-1}).$$

$\Lambda[t]$ is the character of the total exterior power representation which factors through the double cover $Spin(m) \to SO(m)$. As for $\Delta_n[t]$, putting $t = 1$, we have the spin-representation $\Delta_n$ in $RSpin(2n+1)$ and the sum of the half-spin representations $\Delta_n^+ + \Delta_n^- = \Delta_n$ in $RSpin(2n)$ with

$$2^{\epsilon}\Delta_n^2 = 2^{\epsilon} \prod_{i=1}^{n} (z_i + z_i^{-1})^2 = 2^{\epsilon} \prod_{i=1}^{n} (1+z_i^2)(1+z_i^{-2}) = \Lambda[1].$$

Putting $t = -1$, we have $\chi = \Delta_n^+ - \Delta_n^-$ in $RSpin(2n)$.

Let $\epsilon : RSpin(m) \to Z$ be the augmentation which assigns to each representation its dimension, so that $\epsilon(\Lambda^i) = \binom{m}{i}$, $\epsilon(\Delta_n) = 2^n$, $\epsilon(\Delta_n^{\pm}) = 2^{n-1}$. Later on, when working with the Koszul resolution, it will be preferable to replace the exterior powers by the augmentation zero $K$-Pontrjagin classes

$$\pi[t] = \sum_{i \geq 0} t^i \pi_i = \prod_{i=1}^{n} (1 + t(z_i - z_i^{-1})^2).$$

Note that $t^n$ is the highest power of $t$ occurring in this product; so $\pi_i = 0$ for $i > n$.

One has

$$\Lambda[t] = (1+t)^m \prod_{i=1}^{n} \left(1 + (z_i - z_i^{-1})^2 \frac{t}{(1+t)^2}\right) = (1+t)^m \pi \left[\frac{t}{(1+t)^2}\right].$$

Let $\delta_n = \Delta_n - 2^n$ and $\delta_n^{\pm} = \Delta_n^{\pm} - 2^{n-1}$. We have (see [4] and [11])

**Theorem 1.** *With the above notation,*

*(i) $RSpin(2n+1)$ is the polynomial ring $Z[\Lambda^1, \cdots, \Lambda^{n-1}, \Delta_n]$ and is also the polynomial ring $Z[\pi_1, \cdots, \pi_{n-1}, \delta_n]$. One has*

$$\Delta_n^2 = \Lambda^n + \Lambda^{n-1} + \cdots + \Lambda^1 + 1 = \pi_n + 4\pi_{n-1} + 16\pi_{n-2} + \cdots + 2^{2n}.$$

*(ii) $RSpin(2n)$ is the polynomial ring $Z[\Lambda^1, \cdots, \Lambda^{n-2}, \Delta_n^+, \Delta_n^-]$ and is also the polynomial ring $Z[\pi_1, \cdots, \pi_{n-2}, \delta_n^+, \delta_n^-] = Z[\pi_1, \cdots, \pi_{n-2}, \chi, \delta_n^+]$. One has*

$$\Delta_n^2 = \pi_n + 4\pi_{n-1} + 16\pi_{n-2} + \cdots + 2^{2n}, \Delta_n^+\Delta_n^- = \Lambda^{n-1} + \Lambda^{n-3} + \Lambda^{n-5} + \ldots$$

*and $\chi^2 = \pi_n$.*

*(iii) Restriction $Res : RSpin(2n+1) \longrightarrow RSpin(2n)$ takes $\Lambda[t]$ to $(1+t)\Lambda[t]$, $\Delta_n$ to $\Delta_n^+ + \Delta_n^-$ , and $\pi[t]$ to $\pi[t]$.*

*(iv) Restriction $Res : RSpin(2n) \longrightarrow RSpin(2n-1)$ takes $\Lambda[t]$ to $(1+t)\Lambda[t]$, $\Delta_n^{\pm}$ to $\Delta_{n-1}$, and $\pi[t]$ to $\pi[t]$.*

## 4. The ring $RH_{m,k}$

Henceforth we shall write $H$ instead of $H_{m,k}$ when $m$ and $k$ are clear from the context.

We next calculate $RH$ in terms of $R\Omega$ and $RSpin(m-k)$. Recall that $m = 2n$ or $2n+1$. We let $m-k = 2c$ or $2c+1$ and $s = n-c$. Thus, for $m = 2n$, $k = 2s$ or $2s-1$, whereas, for $m = 2n+1$ we have $k = 2s$ or $2s+1$. Note that $c = [(m-k)/2]$ always.

4.1. *RH* **for** $m-k$ **odd.** If $m = 2n$ is even, set $\omega = e_1 e_2 \cdots e_m$ as in §2 and define $\tilde{\omega} \in T^n$ to be $(i, \cdot_{(n)} \cdot, i)$ . If $m = 2n+1$ is odd (so $k = 2s$ is even), set $\omega = e_1 \cdots e_k$ as in §2 and define $\tilde{\omega} \in T^n$ to be $(i, \cdot_{(s)} \cdot, i, 1, ..., 1)$. In each case $\tau(\tilde{\omega}) = \omega$. Let $\tilde{\Omega}$ be the subgroup generated by $\tilde{\omega}$. Then $\tilde{\Omega} \approx Z/4$. Since $\omega$ lies in the center of $H$, multiplication in $H$ induces an epimorphism $\tilde{\Omega} \times Spin\,(2c+1) \to H$ which extends the inclusion $Spin(2c+1) \subset H$ and takes $\tilde{\omega}$ to $\omega$, with kernel

$$K = \{(\tilde{\omega}^i, x) : \omega^i = 1\} = \begin{cases} \{(1,1), (\tilde{\omega}^2, 1)\} & \text{if } \omega^2 = 1, \\ \{(1,1), (\tilde{\omega}^2, -1)\} & \text{if } \omega^2 = -1. \end{cases}$$

This induces an inclusion $RH \subset R(\tilde{\Omega} \times Spin(2c+1)) = R\tilde{\Omega} \otimes RSpin(2c+1)$, and we can identify a representation in $R\tilde{\Omega} \otimes RSpin\,(2c+1)$ as coming from $RH$ if it is trivial on $K$.

Now, $R\tilde{\Omega} = R(Z/4) = Z[\phi]/(\phi^4 = 1)$, where $\phi(\tilde{\omega}) = i$, and $R(H/Spin(2c+1)) = R(Z/2) = Z[\theta]/(\theta^2 - 1)$. The composite $\tilde{\Omega} \xrightarrow{\tau} H \longrightarrow H/Spin(2c+1)$ identifies $\theta$ with $\phi^2$. When working in $R(\tilde{\Omega} \times Spin(2c+1))$ we will often drop the symbol $\otimes$ and write for example, $\phi,\ \rho,\ \phi\rho$ instead of $\phi \otimes 1,\ 1 \otimes \rho,\ \phi \otimes \rho$. Note that $RH$ contains $\theta$ coming from $H \to H/Spin(2c+1)$ as well as the exterior powers $\Lambda^j = 1 \otimes \Lambda^j$, since $\Lambda^j(-1) = I$. As for $\Delta_c$, we have $\Delta_c(-1) = -I$. Thus for $\omega^2 = 1$ the element $1 \otimes \Delta_c$ lies in $RH$ and for $\omega^2 = -1$ the element $\phi \otimes \Delta_c$ lies in $RH$. In either case, we refer to this element as $\bar{\Delta}_c$ (note that $\bar{\Delta}_c = \phi^n \otimes \Delta_c$ when $m = 2n$, whereas $\bar{\Delta}_c = \phi^s \otimes \Delta_c$ when $m = 2n+1$ and $k = 2s$).

**Proposition 1.** *Let be $m-k$ odd ($m-k = 2c+1$, $c \geq 0$). Then $RH$ is the subring of $R\tilde{\Omega} \otimes RSpin(2c+1)$ generated by the elements $\theta, \Lambda^1, ..., \Lambda^{c-1}$ and $\bar{\Delta}_c$.*

*Proof.* We have seen above that $RH$ contains the elements listed above. Conversely, arguing as on p.235 of [4], we prove that $RH$ is the subring fixed by a certain automorphism of $R\tilde{\Omega} \otimes RSpin(2c+1)$. The element $(\tilde{\omega}^2, \omega^2)$ lies in the center of $\tilde{\Omega} \otimes Spin(2c+1)$, and so translation by $(\tilde{\omega}^2, \omega^2)$ gives rise to a ring involution $(\tilde{\omega}^2, \omega^2)^*$on $R(\tilde{\Omega} \otimes Spin(2c+1))$ which takes a typical irreducible representation $\rho$ to $\pm\rho$, the sign depending on whether $\rho(\tilde{\omega}^2, \omega^2)$ is $\pm I$ . Since $(\tilde{\omega}^2, \omega^2)$ maps to 1 in $H$, this involution fixes $RH$ elementwise and corresponds to $(\tilde{\omega}^2)^* \otimes (\omega^2)^*$ on $R\tilde{\Omega} \otimes RSpin(2c+1)$. The latter fixes $\theta, \Lambda^1, \cdots, \Lambda^{c-1}, \bar{\Delta}_c$ and takes $\phi$ to $-\phi$. Hence $Fix((\tilde{\omega}^2)^* \otimes (\omega^2)^*)$ is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. It follows that $RH = Fix((\tilde{\omega}^2)^* \otimes (\omega^2)^*)$ and therefore that $RH$ is generated by the elements listed above. □

4.2. *RH* **for** $m$ **and** $k$ **both even.** There is a similar result when both $m$ and $k$ are even. In this case $\omega = e_1 \cdots e_m$ and we define $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$ in $RH$ to be $1 \otimes \Delta_c^+$ and $1 \otimes \Delta_c^-$ respectively if $\omega^2 = +1$, and $\phi \otimes \Delta_c^+$ and $\phi \otimes \Delta_c^-$ respectively if $\omega^2 = -1$.

**Proposition 2.** *Let $m$ and $k$ both be even $(m-k=2c>0)$. Then $RH$ is the subring of $R\tilde{\Omega}\otimes RSpin(2c)$ generated by $\theta,\Lambda^1,\cdots,\Lambda^{c-2}$, $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$.*

*Proof.* As before, the elements listed above belong to $RH$. Conversely, arguing as in Proposition 1, one has that $Fix((\tilde{\omega}^2)^*\otimes(\omega^2)^*)$ is generated by $\theta,\Lambda^1,\cdots,\Lambda^{c-2}$, $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$, as we wanted. □

4.3. *$RH$* **for** $m$ **and** $k$ **both odd.** Let $m$ and $k$ be odd, so that $m-k=2c$, $c>0$. In this case $X_{m,k}$ is not orientable and $H_{m,k}$ cannot be expressed as a quotient of $\tilde{\Omega}\times Spin(2c)$. However $RH_{m,k}$ can be calculated in terms of $RSpin(2c)$ and $RH_{m,k+1}$. Since $m-(k+1)=2c-1$ is odd, $RH_{m,k+1}$ is known from Proposition 1. The argument we use is based on the calculation of $RO(2n)$ at the end of [4]. Recall that $\omega$ is now $e_1\cdots e_{k+1}$ , an element in the center of $H_{m,k+1}$ (but not of $H_{m,k}$). There is a commutative diagram of restriction homomorphims

$$\begin{array}{ccccc} & & RSpin(2c) & & \\ & \overset{Res_s}{\nearrow} & & \overset{Res}{\searrow} & \\ RH_{m,k} & & & & RSpin(2c-1) \\ & \underset{Res_0}{\searrow} & & \underset{Res}{\nearrow} & \\ & & RH_{m,k+1} & & \end{array}$$

*Remark* 1. The reason for choosing the subgroups $Spin(2c)$ and $H_{m,k+1}$ is that, if $T(c)$ and $T(c-1)$ are "the" maximal tori of $Spin(2c)$ and $Spin(2c-1)$ respectively, then $T(c)$ and $\Omega T(c-1)=\Omega\times T(c-1)$ are Cartan subgroups of $H_{m,k}$ contained in $Spin(2c)$ and $H_{m,k+1}$ respectively.

Now $H_{m,k}$ has two connected components and so possesses exactly two conjugacy classes of Cartan subgroups. From this it follows (see [4]) that the homomorphism

$$(Res_s,Res_0):RH_{m,k}\longrightarrow RSpin(2c)\oplus RH_{m,k+1}\subset RT(c)\oplus R(\Omega T(c-1))$$

is a monomorphism. We will use the above to calculate $RH_{m,k}$ in terms of generators and relations. The generators in question are $\theta,\Lambda^1,\cdots,\Lambda^{c-1}$ and $\bar{\Delta}_c$, defined as follows. The element $\theta$ comes from $H_{m,k}\to H_{m,k}/Spin(2c)$, and $\Lambda^1,\cdots,\Lambda^{c-1}$ are the exterior power coming from $H_{m,k}\to H_{m,k}/\pm1$, where $H_{m,k}/\pm1=K_{m,k}\supset SO(2c)$. Finally, $\bar{\Delta}_c$ is the representation of $H_{m,k}$ induced from $\Delta_c^+$ (or equivalently $\Delta_c^-$). The elements $\theta,\Lambda^1,\cdots,\Lambda^{c-1}$, $\bar{\Delta}_c$ of $RH_{m,k}$ restrict to $1,\Lambda^1,\cdots,\Lambda^{c-1},\Delta_c$ in $RSpin(2c)$, and to $\theta,\Lambda^1+\theta\Lambda^0,\cdots,\Lambda^{c-1}+\theta\Lambda^{c-2}$ and $(1+\theta)\bar{\Delta}_{c-1}$ in $RH_{m,k+1}$ (Lemma 1 below).

**Proposition 3.** *Let $m$ and $k$ both be odd $(m-k=2c>0)$. Then $RH$ is generated by the elements $\theta,\Lambda^1,\cdots,\Lambda^{c-1}$, $\bar{\Delta}_c$ (defined above), subject only to the relations $\theta^2=1$ and $\theta\bar{\Delta}_c=\bar{\Delta}_c$. Furthermore, $\Lambda^{2c-i}=\theta\Lambda^i$ and $\Lambda[1]=\bar{\Delta}_c^2$.*

*Proof.* This is divided up into various lemmas. □

**Lemma 1.** *$Res_0(\bar{\Delta}_c)=(1+\theta)\bar{\Delta}_{c-1}$ and $Res_0(\Lambda^i)=\Lambda^i+\theta\Lambda^{i-1}$.*

*Proof.* By definition $Res_s(\bar{\Delta}_c)=\Delta_c^++\Delta_c^-$, so the restriction of $\bar{\Delta}_c$ to $Spin(2c-1)$ is $2\Delta_{c-1}$. From the definition of $\bar{\Delta}_c$ as an induced representation one has that, on $\Omega T(c-1)$, trace $\bar{\Delta}_c$ is zero. But $(1+\theta)\bar{\Delta}_{c-1}\in RH_{m,k+1}$ also restricts to $2\Delta_{c-1}$ on $RSpin(2c-1)$ and has trace zero on $\Omega T(c-1)$. Hence, $Res_0(\bar{\Delta}_c)=(1+\theta)\bar{\Delta}_{c-1}$. As for $\Lambda^i$, we have that, on $\Omega T(c-1)$, the trace of $\Lambda^i$ is the $i$th symmetric polynomial on $z_1^2, z_1^{-2}, \cdots, z_{c-1}^2, z_{c-1}^{-2}, 1$ plus the $(i-1)$st symmetric polynomial on $z_1^2\theta,\ z_1^{-2}\theta\ , \cdots, z_{c-1}^2\theta,\ z_{c-1}^{-2}\theta,\ \theta$. Hence $Res_0(\Lambda^i) = \Lambda^i + \theta\Lambda^{i-1}$. $\square$

**Corollary 1.** *In $RH_{m,k}$ one has $\theta\bar{\Delta}_c = \bar{\Delta}_c$ and $\Lambda^{2c-i} = \theta\Lambda^i$.*

*Proof.* The first equation follows from the fact that $Res_0(\theta\bar{\Delta}_c) = \theta(1+\theta)\bar{\Delta}_{c-1} = Res_0(\bar{\Delta}_c)$ (since $\theta^2 = 1$ implies $\theta\,(1+\theta) = 1+\theta$) and $Res_s(\theta\bar{\Delta}_c) = Res_s(\bar{\Delta}_c)$. The second equation follows from the fact that $Res_s(\theta\Lambda^i) = \Lambda^i \approx \Lambda^{2c-i} = Res_s(\Lambda^{2c-i})$ and $Res_0(\theta\Lambda^i) = \theta(\Lambda^i + \theta\Lambda^{i-1}) = \theta\Lambda^i + \Lambda^{i-1} \approx \theta\Lambda^{(2c-1)-i} + \Lambda^{(2c-1)-(i-1)} \approx \Lambda^{2c-i} + \theta\Lambda^{2c-i-1} = Res\ _0(\Lambda^{2c-i})$, so that $\theta\Lambda^i \approx \Lambda^{2c-i}$. $\square$

We next identify $Res_s(RH_{m,k})$ and $Res_0(RH_{m,k})$ as fixed rings of certain automorphisms of $RSpin(2c)$ and $RH_{m,k}$. To this end, consider the elements $\omega$ in the center of $RH_{m,k+1}$ and $\alpha = e_{k+1}\cdots e_m$ in the center of $Spin(2c)$. Then $\omega$ normalizes $Spin(2c)$ and $\alpha$ normalizes $H_{m,k+1}$, since $\alpha$ anticommutes with $\omega$.

Let $C_\omega : RSpin(2c) \to RSpin(2c)$ be the automorphism induced by conjugation by $\omega$, and similarly for $C_\alpha : RH_{m,k+1} \to RH_{m,k+1}$. Clearly $C_\omega$ fixes anything in the image of $Res_s$, since $C_\omega : RH_{m,k} \to RH_{m,k}$ is the identity. Hence $C_\omega$ fixes $\Lambda^1, \cdots, \Lambda^{c-1}$. As for $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$, we note that $\omega$ normalizes the maximal torus $T(c)$ of $Spin(2c)$, and so gives rise to $C_\omega : RT(c) \to RT(c)$. It is easy to see that $C_\omega$ inverts just one $z_j$ (namely $z_c$) and so exchanges $\Delta_c^+$ and $\Delta_c^-$.

Turning now to $C_\alpha : RH_{m,k+1} \to RH_{m,k+1}$, we see that $C_\alpha$ fixes the elements $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$ in $Res_0(RH_{m,k})$. Since $RH_{m,k+1}$ is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-2}$ and $\bar{\Delta}_{c-1}$ we need only look at $C_\alpha(\bar{\Delta}_{c-1})$.

**Lemma 2.** $C_\alpha(\bar{\Delta}_{c-1}) = \theta\bar{\Delta}_{c-1}$.

*Proof.* Since $\alpha$ belongs to the center of $Spin(2c)$, it follows that on $Spin(2c-1)$ we have $C_\alpha(\bar{\Delta}_{c-1}) = \bar{\Delta}_{c-1} = \theta\bar{\Delta}_{c-1}$. And, for $h \in Spin(2c-1)$, we have

$$\begin{aligned} C_\alpha(\bar{\Delta}_{c-1})(\omega h) &= \bar{\Delta}_{c-1}(\alpha^{-1}\omega h\alpha) = \bar{\Delta}_{c-1}(-\omega h) = \bar{\Delta}_{c-1}(-1)\bar{\Delta}_{c-1}(\omega h) \\ &= -\bar{\Delta}_{c-1}(\omega h) = \theta(\omega h)\bar{\Delta}_{c-1}(\omega h), \end{aligned}$$

proving the lemma. $\square$

Let $A$ be the subring of $RH_{m,k}$ generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. Our objective is to prove (Lemma 4) that $RH_{m,k} = A$.

**Lemma 3.** *$Res_s(RH_{m,k}) = Res_s(A)$ and $Res_0(RH_{m,k}) = Res_0(A)$.*

*Proof.* First, $Res_s(A) \subseteq Res_s(RH_{m,k}) \subseteq Fix(C_\omega)$. Conversely, $RSpin(2c) = Z[\Lambda^1, \cdots, \Lambda^{c-2}, \Delta_c^+, \Delta_c^-]$, and $C_\omega$ fixes $\Lambda^1, \cdots, \Lambda^{c-2}$ and swaps $\Delta_c^+$ and $\Delta_c^-$. Hence $Fix(C_\omega)$ is generated by $\Lambda^1, \cdots, \Lambda^{c-2}, \Delta_c^+\Delta_c^-$ and $\Delta_c^+ + \Delta_c^-$, i.e., by $\Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. But the latter all belong to $Res_s(A)$. Hence $Res_s(RH_{m,k}) = Res_s(A)$, as required.

Next note that, as a $Z[\theta, \Lambda^1, \cdots, \Lambda^{c-1}]$-module, $RH_{m,k+1}$ is freely generated by 1 and $\bar{\Delta}_{c-1}$ since $\bar{\Delta}_{c-1}^2$ lies in $Z[\theta, \Lambda^1, \cdots, \Lambda^{c-1}]$ (cf.§3, Theorem 1*(i)*). Since $C_\alpha$ fixes $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and sends $\bar{\Delta}_{c-1}$ to $\theta\bar{\Delta}_{c-1}$ (Lemma 2), it follows that $Fix(C_\alpha)$ is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$ . Since $\theta$ and $\Lambda^i + \theta\Lambda^{i-1}$ both lie in $Res_0(RH_{m,k})$, so does $\Lambda^i$. Hence $Res_0(RH) = Res_0(A)$. $\square$

**Lemma 4.** $A = RH_{m,k}$

*Proof.* Let $x \in RH_{m,k}$. By Lemma 3 there exists $y \in A$ such that $Res_s(x) = Res_s(y)$. So we may assume that $Res_s(x) = 0$. Now consider $Res_0(x)$. It belongs to $Res_0(RH_{m,k})$ which by Lemmas 1 and 3 is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$. Since $((1+\theta)\bar{\Delta}_{c-1})^2 \in Z[\theta, \Lambda^1, \cdots, \Lambda^{c-1}]$, we may write $Res_0(x)$ in the form $a + b\theta + (1+\theta)d\bar{\Delta}_{c-1}$, where $a, b, d \in Z[\Lambda^1, \cdots, \Lambda^{c-1}]$. But $ResRes_0(x) = ResRes_s(x) = 0$ (where in both cases $Res$ stands for "restriction to $Spin(2c-1)$"). Hence $a + b + 2d\Delta_{c-1} = 0$ in $RSpin(2c-1)$. Now $RSpin(2c-1) = Z[\Lambda^1, \cdots, \Lambda^{c-2}, \Delta_{c-1}]$ is freely generated over $Z[\Lambda^1, \cdots, \Lambda^{c-1}]$ by 1 and $\Delta_{c-1}$. It follows that $a + b = 0$ and $d = 0$. Thus, $Res_0(x) = (1-\theta)a$. Now, $Res_0 : RH_{m,k} \longrightarrow RH_{m,k+1}$ sends $\Lambda^1, \cdots, \Lambda^{c-1}$ to $\Lambda^1 + \theta, \cdots, \Lambda^{c-1} + \theta\Lambda^{c-2}$ and hence sends $(1-\theta)\Lambda^1, \cdots$ to $(1-\theta)(\Lambda^1+\theta), \cdots$, which is to say $(1-\theta)(\Lambda^1 - 1), \cdots$, since $\theta^2 = 1$. Thus $Res_0$ sends $(1-\theta)\tilde{a}$ (where $\tilde{a} = \tilde{a}(\Lambda^1, \cdots, \Lambda^{c-1})$ in $RH_{m,k}$) into $(1-\theta)\tilde{a}(\Lambda^1 - 1, \cdots, \Lambda^{c-1} - \Lambda^{c-2})$. Now $\tilde{a}$ may be chosen so that $Res_0((1-\theta)\tilde{a}) = (1-\theta)a$. But clearly $Res_s((1-\theta)\tilde{a}) = 0$. Recalling that $Res_s(x) = 0$, we conclude that $x = (1-\theta)\tilde{a}$, since both restrict to the same elements in $RSpin(2c)$ and $RH_{m,k+1}$. Thus $RH_{m,k} = A$, since $(1-\theta)\tilde{a} \in A$. □

Finally we determine the relations in $RH$.

**Lemma 5.** *All relations among $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ in $RH$ are consequences of $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$.*

*Proof.* We know that $\theta^2 = 1$, $\theta\bar{\Delta}_c = \bar{\Delta}_c$ and $\theta\Lambda^i = \Lambda^{2c-i}$ do hold in $RH$ (Corollary 1). Suppose we have a relation among $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$,and $\bar{\Delta}_c$. Using $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$ we may write this as $f(\bar{\Delta}_c) + a\theta = 0$, where $f \in Z[\Lambda^1, \cdots, \Lambda^{c-1}][x]$ and $a \in Z[\Lambda^1, \cdots, \Lambda^{c-1}]$. Then $0 = Res_s(0) = f(\bar{\Delta}_c) + a$. But $\Lambda^1, \cdots, \Lambda^{c-1}, \Delta_c$ are algebraically independent in $RSpin(2c)$. Thus $f(x) = -a$ in $Z[\Lambda^1, \cdots, \Lambda^{c-1}][x]$. So, the relation is $(1-\theta)a = 0$. But then $0 = Res_0(0) = (1-\theta)Res_0(a)$. Now $a$ is a polynomial in $\Lambda^1, \cdots, \Lambda^{c-1}$ and as above

$$(1-\theta)Res_0(a) = (1-\theta)a(\Lambda^1 - 1, \cdots, \Lambda^{c-1} - \Lambda^{c-2}).$$

By Proposition 1, we have $a(\Lambda^1 - 1, \cdots, \Lambda^{c-1} - \Lambda^{c-2}) = 0$. Since $\Lambda^1, \cdots, \Lambda^{c-1}$ are polynomial generators of $Z[\Lambda^1, \cdots, \Lambda^{c-1}]$, as are $\Lambda^1 - 1, \cdots, \Lambda^{c-1} - \Lambda^{c-2}$, we may conclude that $a = 0$. So, all relations follow from $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$. This proves Lemma 5.

This also concludes the proof of Proposition 3. □

We collect the above results on the structure of $RH$ in Theorem 2 below. In the case $mk$ even, it will be convenient to introduce Pontrjagin classes in $RH$ analogous to the ones in $RSpin(m)$. The easiest way to do this is to note that, in Proposition 1, the generators $\Lambda^1, \cdots, \Lambda^{c-1}, \theta$, and $\bar{\Delta}_c$ may be replaced by $\pi_1, \cdots, \pi_{c-1}$, $y = \theta - 1$, and $\delta_c$ and similarly in Proposition 2, as in §3. When $m$ and $k$ are odd, Pontrjagin classes are not available; this is one of the reasons why this case turns out to be much thornier than the rest; but since we will use the Koszul resolution later on for all cases, we use the augmentation zero classes $\lambda_i = \Lambda^i - \binom{2c}{i}, i = 1, \cdots, 2c$, when $mk$ is odd.

**Theorem 2.** *(i) For $m - k = 2c+1, c > 0$ ($k = 2s$ or $k = 2s-1$), $RH$ is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ (or equivalently by $y = \theta - 1, \pi_1, \cdots, \pi_{c-1}$, and $\delta_c$) subject only to the relation $\theta^2 - 1$ (or equivalently $y^2 + 2y = 0$). The element $\pi_c$ is given* *in terms of the generators by*

$$\bar{\Delta}_c^2 = \theta^s \sum_{i=o}^{c} 2^{2(c-i)}\pi_i \qquad \text{if } m \text{ is odd,}$$
$$\bar{\Delta}_c^2 = \theta^n \sum_{i=o}^{c} 2^{2(c-i)}\pi_i \qquad \text{if } m \text{ is even.}$$

*(ii) For $m$ and $k$ both even ($m-k=2c>0, m=2n$ and $k=2s$), $RH$ is generated by $\Lambda^1, \cdots, \Lambda^{c-2}$, $\bar{\Delta}_c^+, \bar{\Delta}_c^-$ and $\theta$ (or equivalently by $\pi_1, \cdots, \pi_{c-2}, \delta_c^+, \delta_c^-$ and $y$) subject only to the relation $\theta^2 = 1$ (equivalently $y^2+2y=0$). The elements $\pi_c$ and $\pi_{c-1}$ are given in terms of the generators by*

$$\bar{\Delta}_c^2 = \theta^n \sum_{i=0}^{c} 2^{2(c-i)}\pi_i, \qquad \bar{\Delta}_c^+\bar{\Delta}_c^- = \theta^n \sum_{i=0}^{c-1} 2^{2(c-1-i)}\pi_i,$$
$$\chi^2 = (\delta_c^+ - \delta_c^-)^2 = \theta^n \pi_c.$$

*(iii) For $m$ and $k$ both odd ($m-k=2c>0$) $RH$ is generated by $\theta, \Lambda^1, \cdots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ (equivalently by $y, \lambda_1, \cdots, \lambda_{c-1}$, and $\delta_c$) subject only to the relations $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$ (equivalently by $y^2+2y=0$ and $\delta_c y = -2^c y$). The element $\Lambda^c$ is given in terms of the generators by*

$$\bar{\Delta}_c^2 = (1+\theta)(1+\Lambda^1+...+\Lambda^{c-1}) + \Lambda^c.$$

*Proof.* The theorem follows almost immediately from Propositions 1, 2 and 3. □

*Remark* 2. In addition to $y^2+2y=0$, notice the following formulae, which follow from the definition of y and $\theta$: $y^j = (-2)^{j-1}y$ for $j>0$, $y\theta^j = y(1+y)^j = (-1)^j y$, for $j \geq 0$, $(1+y)^2 = 1$, $(1+\theta)^s = 2^{s-1}(1+\theta)$, $\theta(1+\theta) = 1+\theta$ and $y(1+\theta) = 0$. These will be used many times henceforth and without specific mention.

## 5. The restriction $Res : RSpin(m) \longrightarrow RH$

5.1. **Restriction for** $m-k$ **odd.** Let $\tilde{\Omega}$ be as in 4.1. We have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{\Omega} \times Spin(2c+1) & \longrightarrow & H \subset Spin(m) \\
\uparrow i & & \uparrow j \\
\tilde{\Omega} \times T(c) & \xrightarrow{\ \mu\ } & T(n) \\
\uparrow Id\times\tau & & \uparrow \tau \\
\tilde{\Omega} \times T^c & \xrightarrow{\ \mu\ } & T_n
\end{array}$$

where $\mu$ stands for multiplication and $i, j$ are inclusions. Since $\omega$ lies in the center of $H$, every element of $H$ is conjugate to an element of $\tilde{\Omega}T(c)$. Hence it suffices to calculate the homomorphism $RSpin(m) \to R\tilde{\Omega} \otimes RT(c)$ induced by $\mu$, or equivalently to calculate $\mu^{\#} : RT^n \longrightarrow R\tilde{\Omega} \otimes RT^c$.

*Case* 1. $k$ even, $m$ odd ($m = 2n+1, k = 2s, s+c = n$).

In this case the restriction is *Res:* $RSpin(2n+1) \to RH_{2n+1,2s}$. Hence

$$Res\colon\ Z[\pi_1, \cdots, \pi_{n-1}, \delta_n] \to Z[\bar{\pi}_1, \cdots, \bar{\pi}_{c-1}, \delta_c, y]/(y^2+2y)$$

Here, $\bar{\pi}[t]$ denotes the $\pi[t]$ corresponding to $RH[t]$. Observing, as in 4.1, that $Z/4 \approx \tilde{\Omega} \subset T^n$ is generated by $\tilde{\omega} = (i, \cdot_{(s)} \cdot, i, 1 \cdots, 1)$ and that a generic element $\xi \in T^c$ has the form $\xi = (1, ..., 1, e^{i\theta_{s+1}}, ..., e^{i\theta_n})$, it is easy to see that

$$\mu^{\#}(z_j) = \begin{cases} \phi, & 1 \le j \le s, \\ z_j, & s+1 \le j \le n. \end{cases}$$

Hence, defining $\pi'[t] = \mu^{\#}\pi[t]$, we have

$$\begin{aligned} \pi'[t] &= \mu^{\#} \prod_{j=1}^{n} (1 + t(z_j - z_j^{-1})^2) \\ &= \prod_{j=1}^{s} (1 + t(\phi - \phi^{-1})^2) \prod_{j=s+1}^{n} (1 + t(z_j - z_j^{-1})^2) \qquad (1) \\ &= (1+2ty)^s \bar{\pi}[t] \quad (\text{since } (\phi - \phi^{-1})^2 = 2\theta - 2 = 2y). \end{aligned}$$

It follows that

$$\bar{\pi}[t] = (1+2ty)^{-s} \pi'[t], \qquad (2)$$

so that $\bar{\pi}_i$ is a linear combination of $\pi'_1, \cdots, \pi'_i$ with coefficients in $Z[y]$, for $i = 1, \cdots, c$.

Next, consider $\Delta_n \in RSpin(2n+1)$. Since $\Delta_n = \prod_{j=1}^{n} (z_j + z_j^{-1})$, we have

$$\begin{aligned} j^{\#}(\Delta_n) &= \prod_{j=1}^{s} (\phi + \phi^{-1}) \prod_{j=s+1}^{n} (z_j + z_j^{-1}) \\ &= \left\{ \begin{array}{ll} \phi(1+\theta)^s \Delta_c & \text{if } s \text{ odd} \\ (1+\theta)^s \Delta_c & \text{if } s \text{ even} \end{array} \right\} = (1+\theta)^s \bar{\Delta}_c. \end{aligned}$$

So, we obtain

$$Res(\Delta_n) = 2^{s-1}(2+y)\bar{\Delta}_c. \qquad (3)$$

From (1) and (3), we have

**Proposition 4.** *For $m = 2n+1$ and $k = 2s$, $Res : RSpin(2n+1) \to RH$ is given by*

$$\begin{aligned} Res(\pi[t]) &= \pi'[t] = (1+2ty)^s \bar{\pi}[t], \\ Res(\delta_n) &= 2^{s-1}(2+y)\delta_c + 2^{n-1}y. \end{aligned}$$

*Case* 2. $k$ odd, $m$ even ($m = 2n, k = 2s-1$).

In this case the restriction is *Res:* $RSpin(2n) \to RH_{2n,2s-1}$. Hence

$$Res\colon\ Z[\pi_1, \cdots, \pi_{n-2}, \delta_n^+, \chi_n] \to Z[\bar{\pi}_1, \cdots, \bar{\pi}_{c-1}, \delta_c, y]/(y^2+2y).$$

This time $\tilde{\omega} = (i, \cdot_{(n)} \cdot, i)$ and $\xi = \left(1, \cdots, 1, e^{i\theta_{s+1}}, \cdots, e^{i\theta_n}\right)$ imply

$$\mu^{\#}(z_j) = \begin{cases} \phi, & 1 \le j \le s, \\ \phi z_j, & s+1 \le j \le n. \end{cases}$$

Hence

$$\begin{aligned}\pi'(t) &= \mu^{\#}(\pi[t]) = \mu^{\#}(\prod_{j=1}^{n}[1+t(z_j-z_j^{-1})^2]\\ &= \prod_{j=1}^{s}[1+t(\phi-\phi^{-1})^2]\prod_{j=s+1}^{n}[1+t(\phi z_j-(\phi z_j)^{-1})^2]\\ &= (1+2ty)^s\prod_{j=s+1}^{n}[1+2t(\theta-1)+t\theta(z_j-z_j^{-1})^2]\\ &= (1+2ty)^n\prod_{j=s+1}^{n}\left[1+(z_j-z_j^{-1})^2\frac{t\theta}{(1+2ty)}\right].\end{aligned}$$

Therefore,

$$\pi'[t]=(1+2ty)^n\bar{\pi}[u], \tag{4}$$

where $u=u(t)=t(1+y)/(1+2ty)$. Since $(1+2ty)(1+2uy)=1$ one has $u(u(t))=1$, so that

$$\bar{\pi}[t]=(1+2ty)^n\pi'[u], \tag{5}$$

from which we conclude as before that $\bar{\pi}_i$ is a linear combination of $\pi'_1,\cdots,\pi'_i$ with coefficients in $Z[y]$, $i=1,\cdots,c$.

Next, we have (in $RSpin(2n)$) $\Delta_n^+$ and $\Delta_n^-$. If we write, as in §3, $\Delta_n[t]=\prod_{j=1}^{n}(z_j+tz_j^{-1})$, we have $\Delta_n[t]=\Delta_n^++t\Delta_n^-$ for $t=\pm1$. Hence, for $t=\pm1$,

$$\begin{aligned}j^{\#}(\Delta_n[t]) &= \prod_{j=1}^{s}(\phi+t\phi^{-1})\prod_{j=s+1}^{n}(\phi z_j+t\phi^{-1}z_j^{-1})\\ &= (1+t\theta)^s\phi^n\prod_{j=s+1}^{n}(z_j+t\theta z_j^{-1})\\ &= \phi^n(1+t\theta)^s\Delta_c[t\theta]=2^{s-1}(1+t\theta)\phi^n\Delta_c[t\theta],\end{aligned}$$

from which it follows that

$$Res(\Delta_n^+)=2^{s-1}\bar{\Delta}_c \text{ and } Res(\Delta_n^-)=2^{s-1}\theta\bar{\Delta}_c. \tag{6}$$

From (4) and (6) we have

**Proposition 5.** *For $m=2n$ and $k=2s-1$, $Res$: $RSpin(2n)\to RH$ is given by*

$$Res(\pi[t])=\pi'[t]=(1+2ty)^n\bar{\pi}\left[t(1+y)/(1+2ty)\right],$$
$$Res(\delta_n^+)=2^{s-1}\delta_c,$$
$$Res(\chi_n)=-2^{s-1}\delta_c y-2^{n-1}y.$$

5.2. **Restriction for $m$ and $k$ both even.** Next $m=2n$, $k=2s$ and as before $c=n-s$. The restriction is given by *Res:* $RSpin(2n)\to RH_{2n,2s}$. Hence

$$Res\colon\ Z[\pi_1,\cdots,\pi_{n-2},\delta_n^+,\chi_n]\to Z[\bar{\pi}_1,\cdots,\bar{\pi}_{c-2},\delta_c^+,\delta_c,y]/(y^2+2y).$$

As in 5.1, case 2, we have

$$\pi'[t]=(1+2ty)^n\bar{\pi}[u], \tag{7}$$

$$Res(\Delta_n^+)=2^{s-1}\bar{\Delta}_c \text{ and } Res(\Delta_n^-)=2^{s-1}\theta\bar{\Delta}_c \tag{8}$$

(here $\bar{\Delta}_c = \bar{\Delta}_c^+ + \bar{\Delta}_c^-$). Hence, from (7) and (8) we have

**Proposition 6.** *For $m = 2n$ and $k = 2s$, $Res{:}RSpin(2n) \to RH$ is given by*

$$\begin{aligned} Res(\pi[t]) &= \pi'[t] = (1+2ty)^n \bar{\pi}\left[t(1+y)/(1+2ty)\right], \\ Res(\delta_n^+) &= 2^{s-1}\delta_c, \quad (\delta_c = \delta_c^+ + \delta_c^-), \\ Res(\chi_n) &= -2^{s-1}\delta_c y - 2^{n-1}y. \end{aligned}$$

5.3. **Restriction for $m$ and $k$ both odd.** Here $m = 2n+1,\ k = 2s+1$. We write $\overline{\Lambda}[t]$ for the corresponding $\Lambda[t]$ to $RH_{m,k}[t]$. The restriction is given by $Res{:}RSpin(2n+1) \to RH_{2n+1,2s+1}$. Hence

$$Res : Z[\Lambda^1, \cdots, \Lambda^{n-1}, \Delta_n] \to Z[\bar{\Lambda}^1, \cdots, \bar{\Lambda}^{c-1}, \bar{\Delta}_c, \theta]/(\theta^2 - 1, \theta\bar{\Delta}_c - \bar{\Delta}_c).$$

We may identify $Res(x),\ x \in RSpin(2n+1)$, by calculating the restriction of $x$ to $RSpin(2c)$ and to $RH_{m,k+1}$. In the case of $RH_{m,k+1}$, going back to 5.1, case 1, where $Res{:}RSpin(2n+1) \to RH_{m,k+1}$, we have (note $k+1 = 2(s+1)$) that

$$\begin{aligned} Res(\Lambda[t]) &= \mu^{\#}(1+t)\prod_{i=1}^{n}(1+tz_i^2)(1+tz_i^{-2}) \\ &= (1+t)\prod_{j=1}^{s+1}(1+t\theta)(1+t\theta^{-1}) \prod_{j=s+2}^{n}(1+tz_j^2)(1+tz_j^{-2}) \\ &= (1+t\theta)^{2s+2}\hat{\Lambda}[t] \end{aligned}$$

($\hat{\Lambda}[t]$ denotes the $\Lambda[t]$ corresponding to $RH_{m,k+1}$) and $Res(\Delta_n) = 2^s(\theta+1)\bar{\Delta}_{c-1}$.

Now, $\Lambda[t]$ restricts to $(1+t)^{2s+1}\Lambda[t]$ in $RSpin(2c)[t]$, and from 4.3 we have that $Res_0(\Lambda[t]) = (1+t\theta)\hat{\Lambda}[t]$ and $Res_s(\bar{\Lambda}[t]) = \Lambda[t] \in RSpin(2c)[t]$. Hence

$$Res(\Lambda[t]) = \Lambda'[t] = (1+t\theta)^k\bar{\Lambda}[t]. \tag{9}$$

The element $\Delta_n \in RSpin(2n+1)$ restricts to $2^s\Delta_c \in RSpin(2c)$. But $Res_s(\bar{\Delta}_c) = \Delta_c$ and $Res_0(\bar{\Delta}_c) = (1+\theta)\bar{\Delta}_{c-1}$. Hence,

$$Res(\Delta_n) = 2^s\bar{\Delta}_c. \tag{10}$$

From (9) and (10) we have

**Proposition 7.** *For $m = 2n+1,\ k = 2s+1$, $Res{:}RSpin(2n+1) \to RH$ is given by*

$$\begin{aligned} Res(\Lambda[t]) &= \Lambda'[t] = (1+t\theta)^k\bar{\Lambda}[t], \\ Res(\Delta_n) &= 2^s\bar{\Delta}_c. \end{aligned}$$

## 6. The Hodgkin spectral sequence

6.1. **The theorem.** Let $G$ be a compact, connected Lie group with $\pi_1(G)$ torsion-free and $H$ a closed subgroup of $G$. The Hodgkin spectral sequence calculates the complex $K$-theory of $G/H$ in terms of $RG$ and $RH$. The result is the following (cf. [10], [12]).

**Theorem 3.** *Given $G$ and $H$ as above, there is a strongly convergent sequence $E_r(G/H)$ with the following three properties:*
*(i) As an algebra, $E_2^p(G/H) = Tor_{RG}^p(RH; Z)$,*
*(ii) The differential $d_r : E_r^{p-r} \to E_r^p$ is zero for $r$ even,*

*(iii)* $E_\infty^*(G/H)$ *is the graded algebra associated to a negative filtration of* $K^*(G/H)$ *compatible with its multiplication*

$$F^pK^* \otimes F^qK^* \to F^{p+q}K^*,$$

$$F^pK^* = F_{-p}K^* = \tilde{F}^{2p}K^0 \oplus \tilde{F}^{2p+1}K.$$

**Notes**: 1. $RG, RH$ and $R\{1\} = Z$ are trivially $Z/2$-graded by $R^0 = R$ and $R^1 = 0$.

2. If $Tor_{RG}^*(RH, \mathrm{Z})$ is generated as an algebra by its elements of degree $\leq 2$, then the Hodgkin spectral sequence collapses. Hence in this case we have two isomorphisms

$$E_2^{0,0} = RH \otimes_{RG} Z \xrightarrow{\approx} E_\infty^{0,0} = F_0K^0,$$

$$E_2^{-1,0} = Tor_{RG}^{-1}(RH; Z) \xrightarrow{\approx} E_\infty^{-1,0} = F_{-1}K^{-1}.$$

6.2. **Some consequences.** Let $G$ and $H$ be as above, and let *Res*: $RG \to RH$ be the homomorphism induced by inclusion.

As before, $\epsilon_G : RG \to R\{1\} = Z$ is the augmentation which assigns to each representation its dimension. We will suppose that $RG = Z[\gamma_1, \cdots, \gamma_n]$ with $\epsilon_G(\gamma_i) = 0$, $i = 1, \cdots, n$ (if $\epsilon_G(\gamma_i) \neq 0$ we take $\tilde{\gamma}_i = \gamma_i - \epsilon_G(\gamma_i)$), and $RH$ is generated by $h_1, ..., h_m$ with $\epsilon_H(h_j) = 0$, $j = 1, \cdots, m$.

Let $\Gamma = Z[\gamma_1, \cdots, \gamma_r], 1 \leq r \leq m$. We state the following straightforward applications of the change of rings theorem and Koszul resolution (cf. [5] and [12]):

1. If $RH$ is a free (or more generally flat) $\Gamma$-module and $Res(\gamma_i) = h_i, i = 1, \cdots, r$, it follows that $Tor_{RG}^*(RH; Z) = Tor_A^*(B; Z)$, where

$$A = RG/(\gamma_1, \cdots, \gamma_r), \qquad B = RH/(h_1, \cdots, h_r)$$

   and the $A$-module structure of $B$ (which we call $Res/A$, or *Res* by a small abuse of notation) is induced by *Res*.

2. If $\tau_1, \cdots, \tau_s \in A$ and $(Res/A)(\tau_i) = 0, 1 \leq i \leq s$, then, setting $A' = A/(\tau_1, \cdots, \tau_s)$, $B$ is an $A'$-module via $Res/A$ and

$$Tor_A^*(B; Z) \approx \Lambda_Z^*[t_1, \cdots, t_s] \otimes Tor_{A'}^*(B; Z),$$

   where $\Lambda_Z^*[t_1, \cdots, t_s]$ denotes the graded exterior algebra with generators $t_i$ of degree 1,

3. If $RH$ is a trivial $RG$-module, then

$$Tor_{RG}^*(RH; Z) \approx \Lambda_{RG}^*[t_1, \cdots .t_n],$$

   where $t_1, \cdots, t_n$ are of degree 1.

## 7. The complex $K$-theory of $X_{m,k}$ for $mk$ even

Now that $RH$ and the restriction *Res:RSpin*$(m) \to RH$ are known, we are in position to determine $K^*(X_{m,k})$ for $mk$ even. To do this we must determine the structure of the ring $B$ defined in §6, as well as the restriction $A \to B$, so as to be able to apply the Koszul resolution.

7.1. **Structure of $B$.** From §5 relations (1), (2), (4), (5) and (7) (cases where $mk$ is even), we see that $\pi'_1, \cdots, \pi'_c$ can be taken as generators for $RH_{m,k}$ instead of $\bar{\pi}_1, \cdots, \bar{\pi}_c$. Since $Res(\pi_i) = \pi'_i, i = 1, \cdots, c$ and $RH$ is a free $Z[\pi_1, \cdots, \pi_c]$-module it follows from 6.2 (1) that

$$Tor^*_{RSpin(m)}(RH; Z) \approx Tor^*_A(B; Z), \tag{11}$$

where

$$A = RSpin(m)/(\pi_1, \cdots, \pi_c), \quad B = RH/(\pi'_1, \cdots, \pi'_c)$$

(unfortunately, for $m$ and $k$ both odd $RH$ is not a free $Z[\lambda_1, \cdots, \lambda_c]$-module).

The homomorphism *Res* induces an algebra homomorphism $Res/A : A \to B$ that makes $B$ an $A$-algebra. While $A$ is still a polynomial algebra (on $c$ fewer generators), the structure of $B$ is more complicated as we now see.

From §5, (1), (5) and (7) we have

$$\bar{\pi}[t] = \begin{cases} (1+2ty)^{-s}\pi'[t] & (m = 2n+1), \\ (1+2ty)^n \pi'[u] & (m = 2n), \end{cases} \tag{12}$$

where $u = t(1+y)/(1+2ty)$.

Now $\bar{\pi}_i = 0$ for $i > c$, and in $B$, we have $\pi'_i = 0$ for $1 \le i \le c$ ($\pi'_0 = 1$). So, for $1 \le i \le c$, in $B$ we have:

$$\bar{\pi}[t] = \begin{cases} T_c(1+2ty)^{-s} & (m = 2n+1), \\ T_c(1+2ty)^n & (m = 2n), \end{cases} \tag{13}$$

where $T_c$ is truncation defined by $T_c(\sum_{i \ge 0} a_i t^i) = \sum_{i=0}^{c} a_i t^i$. Thus,

$$\bar{\pi}_i = \begin{cases} (-1)^{i-1}\binom{n}{i}2^{2i-1}y, \\ -\binom{s+i-1}{i}2^{2i-1}y, \end{cases} \qquad 1 \le i \le c, \tag{14}$$

recalling that $y^2 = -2y$ and the well known identity $\binom{-s}{i} = (-1)^i\binom{s+i-1}{i}$ (cf. [8]).

Substituting $u$ for $t$ in the second equation of (12) and using (4), we get

$$\pi'[t] = (1+2ty)^n \sum_{i \ge c} \binom{n}{i}(2uy)^i. \tag{15}$$

We write $\bar{f}(t)$ for $f(-t/(1+t))$. Observe that $\overline{fg} = \bar{f}\,\bar{g}$ and $\overline{f+g} = \overline{f} + \overline{g}$. Then, $f(2uy) = \overline{f}(2ty)$. Hence $\pi'[t] = F(2ty)$, where $F(t) = (1+t)^n \overline{T_c(1+t)^n}$.

**Lemma 6.** *With the above notation*

*(i)* $(1+t)^s T_c(1+t)^{-s} = (1+t)^n \overline{T_c(1+t)^n}$,

*(ii)* $(1+t)^s T_c(1+t)^{-s} = (-1)^c \sum_{i=0}^{c} \binom{n}{i}\binom{i-1}{c} t^i$.

*Proof.* Fix $s \ge 0$. Both results are proved by induction on $c$ starting with $c = 0$ where both sides of the equations are $(1+t)^s$, since $c = 0$ implies $n = s$. The inductive steps in *(i)* and *(ii)* are obtained by using the elementary identities:

$$\binom{i+1}{j} = \binom{i}{j} + \binom{i}{j-1}, \qquad \binom{i}{j} = (-1)^j \binom{j-i-1}{j},$$
$$\binom{n}{i}\binom{i}{j} = \binom{n}{j}\binom{n-j}{i-j}$$

valid for all $i$, $j$ and $n \geq 0$ (see [8]). Specifically, for *(i)* the inductive step is

$$\begin{aligned}
&(1+t)^{n+1}\overline{T_{c+1}(1+t)^{n+1}} - (1+t)^n\overline{T_c(1+t)^n} \\
&= (1+t)^{n+1}\overline{(T_c(1+t)^{n+1} - (1+t)T_c(1+y)^n)} \\
&= (1+t)^{n+1}\overline{\binom{n}{c+1}t^{c+1}} \text{ (since } tT_c(f) = T_{c+1}(tf)) \\
&= (1+t)^{n+1}\binom{n}{c+1}\left(\frac{-t}{1+t}\right)^{c+1} \\
&= (-1)^{c+1}\binom{n}{c+1}(1+t)^s t^{c+1} = (1+t)^s\binom{-s}{c+1}t^{c+1} \\
&= (1+t)^s T_{c+1}(1+t)^{-s} - (1+t)^s T_c(1+t)^{-s},
\end{aligned}$$

whereas for *(ii)* the inductive step is: consider

$$(-1)^{c+1}\sum_{i=0}^{n}\binom{n}{i}\binom{i-1}{c}t^i - (-1)^c\sum_{i=0}^{n}\binom{n}{i}\binom{i-1}{c}t^i.$$

Using the elementary identities above we have

$$\binom{n}{i}\binom{i-1}{c} + \binom{n+1}{i}\binom{i-1}{c+1} = (-1)^{c+1}\binom{-s}{c+1}\binom{s}{i-c-1}$$

which is equal to the coefficient of $t^i$ in $(-1)^{c+1}\binom{-s}{c+1}(1+t)^s t^{c+1}$. □

Applying Lemma 6 to the expression (15) for $\pi'[t]$ for $m$ even and (13) for $m$ odd, we conclude that in $B$, for $m$ even or odd,

$$\pi'_i = (-1)^{c+1+i}2^{2i-1}\binom{n}{i}\binom{i-1}{c}y, \qquad i > c. \tag{16}$$

In addition to this explicit formula for $\pi'_i$, $i > c$, it is important to establish certain linear relations among these $\pi'_i$. We do this next.

Since $\bar{\pi}_i = 0, i > c$, from (12) we see that for $i > c$

$$\begin{aligned}
0 &= \sum_{j\geq 0}\pi'_j\binom{n-j}{i-j}(1+y)^j(2y)^{i-j} \\
&= \binom{n}{i}(2y)^i + \theta^i\pi\prime_i + \sum_{j=c+1}^{i-1}\binom{n-j}{i-j}\theta^j(2y)^{i-1}\pi\prime_j.
\end{aligned} \tag{17}$$

Substituting in (13), we get

$$\pi'_i = 2^{2i-1}y\binom{n}{i} - \sum_{j=c+1}^{i-1}\binom{n-j}{i-j}2^{2(i-j)}\pi\prime_j. \tag{18}$$

Now, by substituting the values of $\bar{\pi}_i$ given in (14) in the relations for $\delta_c^2$, $\chi^2$ and $\delta_c^+\delta_c^-$ from Theorem 2, and using elementary binomial identities (see [8] for example) we get the relations

$$\delta_c^2 + 2^{c+1}\delta_c = \begin{cases} 2^{2c-1}y\left[1 + (-1)^{s-1}\binom{n-1}{c}\right] & (m = 2n), \\ 2^{2c-1}y\left[1 + (-1)^{s-1}\binom{n}{c}\right] & (m = 2n+1), \end{cases} \tag{19}$$

(20) $$\chi^2 = (\delta_c^+ - \delta_c^-)^2 = \theta^n \bar{\pi}_c = (-1)^{s-1} 2^{2c-1} y \binom{n}{c} \quad (m \text{ even},\ k \text{ even}),$$

(21) $$\delta_c^+ \delta_c^- + 2^{c-1}\delta_c = 2^{2c-3} y \left[1 + (-1)^s \binom{n-1}{c-1}\right] \quad (m \text{ even},\ k \text{ even}),$$

From (18) and $\delta_c = \delta_c^+ + \delta_c^-$, it follows trivially that

(22) $$\delta_c \delta_c^+ = 2^{2c-3} y \left[1 + (-1)^s \binom{n-1}{c-1}\right] - 2^{c-1}\delta_c + (\delta_c^+)^2 \quad (m \text{ even},\ k \text{ even}).$$

The ring structure of $B$ is thus given by

**Proposition 8.** *Let* $L = \left[1 + (-1)^{s-1}\binom{[(m-1)/2]}{c}\right]$ *and* $M = \left[1 + (-1)^s \binom{n-1}{c-1}\right]$.

*(i) For $m-k$ odd, as an abelian group*

$$B \approx Z \oplus Zy \oplus Z\delta_c \oplus Z\delta_c y$$

*with multiplication given by the table*

| | $y$ | $\delta_c$ |
|---|---|---|
| $y$ | $-2y$ | $\delta_c y$ |
| $\delta_c$ | $\delta_c y$ | $2^{c+1}\delta_c + 2^{2c-1} yL$ |

*As a ring, $B \approx Z[\delta_c, y]/I$, where $I$ is the ideal generated by $y^2+2y$ and $\delta_c^2 + 2^{c+1}\delta_c - 2^{2c-1} yL$.*

*(ii) For $m$ and $k$ both even*

$$B \approx Z \oplus Zy \oplus Z\delta_c \oplus Z\delta_c^+ \oplus Z\delta_c y \oplus Z\delta_c^+ y \oplus Z(\delta_c^+)^2 \oplus Z(\delta_c^+)^2 y$$

*with multiplication given by the table*

| | $y$ | $\delta_c$ | $\delta_c^+$ |
|---|---|---|---|
| $y$ | $-2y$ | $y\delta_c$ | $y\delta_c^+$ |
| $\delta_c$ | $y\delta_c$ | $-2^{c+1}\delta_c + 2^{2c-1} yL$ | $-2^{c-1}\delta_c + 2^{2c-3} yM + (\delta_c^+)^2$ |
| $\delta_c^+$ | $y\delta_c^+$ | $-2^{c-1}\delta_c + 2^{2c-3} yM + (\delta_c^+)^2$ | $(\delta_c^+)^2$ |

*or, $B \approx Z[y, \delta_c, \delta_c^+]/J$, where $J$ is the ideal generated by $y^2+2y$, $\delta_c^2 + 2^{c+1}\delta_c - 2^{2c-1} yL$, and $\delta_c\delta_c^+ + 2^{c-1}\delta_c - 2^{2c-3} yM - (\delta_c^+)^2$.*

*Remark* 3. The remaing products, with $(\delta_c^2)^2$ and $y(\delta_c^2)$ are easily deduced in terms of the abelian group generators for B using the products given above.

7.2. **Structure of $A$.** As for $A$, from relation (16) (and proceeding as in [2]), we can choose a new basis for $A$, namely $\tau_1, \cdots, \tau_{s-3}, \rho_1, \rho_2, \rho_3$, such that $Res(\tau_i) = 0$, $i = 1, \cdots, s-3$, and, for $m = 2n+1$ and $k = 2s$, $Res(\rho_1) = 0$, $Res(\rho_2) = Res(\delta_n)$, $Res(\rho_3) = by$, while for $m = 2n$ and $k = 2s$ or $2s-1$, $Res(\rho_1) = Res(\chi_n)$, $Res(\rho_2) = Res(\delta_n^+)$, $Res(\rho_3) = by$.

In both cases $b = g.c.d.\{2^{2i-1}\binom{n}{i},\ i = c+1, \cdots, [m/2]-2\}$, and we remind the reader that the formulae for $Res(\delta_n), Res(\chi_n)$, and $Res(\delta_n^+)$ were given in §5. Hence, from 6.2(2) we have, for $m = 2n$,

(23) $$Tor^*_A(B; Z) \approx \Lambda^*_Z[t_1, \cdots, t_{s-3}] \otimes_Z Tor^*_{A'}(B; Z)$$

with $\dim t_i = 1$, $i = 1, \cdots, s-3$, and $A' = A/(\tau_1, \cdots, \tau_{s-3})$, while, for $m = 2n+1$,

(24) $$Tor^*_A(B; Z) \approx \Lambda^*_Z[t_1, \cdots, t_{s-2}] \otimes_Z Tor^*_{A''}(B; Z)$$

with $\dim t_i = 1$, $i = 1, \cdots, s-2$, and $A'' = A/(\tau_1, \cdots, \tau_{s-3}, \rho_1)$.

7.3. $K^*(X_{m,k})$ **for** $mk$ **even.** We treat separately the three cases $m$ even, $m$ odd $k$ even and, $m$ odd $k$ odd, noting that in all cases $A$, $A'$, and $A''$ are polynomial algebras so that the Koszul resolution is applicable.

7.3.1. *m even.* $Tor^*_{A'}(B;Z)$ is the homology of the Koszul complex $\Lambda^*_B(x_1,x_2,x_3)$, where $d(x_i) = Res(\rho_i)$, $i = 1,2,3$. As in [2], we take a new basis $u_1,u_2,u_3$ for $\Lambda^*_B(x_1,x_2,x_3)$ with $d(u_1) = 2^\alpha y$, $2^\alpha = g.c.d.\{2^{n-1}, b\}$; $d(u_2) = 2^{s-1}\delta_c$; and $d(u_3) = 0$. Hence for $k = 2s-1$

$$Tor^*_{A'}(B;Z) \approx \Lambda^*_Z[z_1,z_2,z_3,z_4] \otimes Z[y,\delta_c]/I, \tag{25}$$

and for $k = 2s$

$$Tor^*_{A'}(B;Z) \approx \Lambda^*_Z[z_1,z_2,z_3,z_4] \otimes Z[y,\delta_c,\delta_c^+]/J \tag{26}$$

with $y$, $\delta_c$ and $\delta_c^+$ of degree 0 and, $z_1,z_2,z_3,z_4$ of degree 1 given by, $z_1 = (y+2)u_1$, $z_2 = -2^{n+c-2-\alpha}Lu_1 + (\delta_c + 2^{c+1})u_2$, $z_4 = -2^{s-1-r}\delta_c u_1 + 2^{\alpha-r}yu_2$ and $z_3 = u_3$.

For $k = 2s-1$, $I$ is the ideal generated by: $y^2+2y$; $\delta_c^2+2^{c+1}\delta_c-2^{2c-1}yL$; $2^{s-1}\delta_c$; $2^\alpha y$; $(y+2)z_4-2^{s-1-r}\delta_c z_1$; $2^r z_4$, where $r = min\{\alpha, s-1\}$; $z_1y$; $z_1z_4$; $z_2z_4$; $z_2\delta_c - 2^{2c-1+r-\alpha}Lz_4$; and $\delta_c z_4 + 2^{c+1}z_4 - 2^{\alpha-r}yz_2$.

For $k = 2s$, $J$ is the ideal generated by the same elements given in $I$ and the element $\delta_c\delta_c^+ + 2^{c-1}\delta_c - 2^{2c-3}yM - (\delta_c^+)^2$.

Therefore we have

**Theorem 4.** *The Hodgkin spectral sequence for $X_{2n,k}$ collapses, and so, as graded algebras,*

$$K^*(X_{2n,2s-1}) \approx \Lambda^*_Z[t_1,\cdots,t_{s-3},z_1,z_2,z_3,z_4] \otimes Z[y,\delta_c]/I,$$
$$K^*(X_{2n,2s}) \approx \Lambda^*_Z[t_1,\cdots,t_{s-3},z_1,z_2,z_3,z_4] \otimes Z[y,\delta_c,\delta_c^+]/J,$$

*where $I$ and $J$ are the ideals generated by the above elements, except that $z_1z_4$ and $z_2z_4$ are replaced by $z_1z_4+\lambda$, $z_2z_4+\mu$ , for some $\lambda$, $\mu \in Z\,[y,\delta_c]$ if $k$ odd, or by $\lambda$, $\mu \in Z\,[y,\delta_c,\delta_c^+]$ if $k$ even.*

*Proof.* This follows from (11), (18), (23), (24), (25) (or (26) for $k = 2s$), Theorem 1, Note 2 of §6 and §6 of [2]. As for the elements $z_iz_4$, $i = 1,2$, they are in $Tor^2 = E_2^\infty = \tilde{F}^2/\tilde{F}^0$. Since $z_iz_4 = 0$ in $Tor^*$, we have $z_iz_4 \in \tilde{F}^0(K^*) \subset K^0$. Hence, in $K^0$, $z_iz_4 = a_iy + b_i\delta_c + c_i\delta_c y, a_i,b_i,c_i \in Z$, $i = 1,2$ for $k = 2s-1$ (similarly for $k$ even with the extra terms added). □

7.3.2. *m odd, k even.* $Tor^*_{A''}(B;Z)$ is the homology of the Koszul complex $\Lambda^*_B(x_1,x_2)$:

$$0 \longrightarrow B \longrightarrow B\oplus B \xrightarrow{d} B \longrightarrow 0$$

where $d(x_1) = Res(\rho_3) = by$ and $d(x_2) = Res(\delta_n) = 2^{s-1}(2+y)\delta_c + 2^{n-1}y$. Then, in $Tor^0_{A''}(B;Z)$ we have $by = 0$ and $2^{s-1}(2+y)\delta_c + 2^{n-1}y = 0$. Hence, by multiplying the second equation by $y$, it follows that $2^n y = 0$. Thus $2^\alpha y = 0$, where $2^\alpha = g.c.d.\{2^n, b\}$.

For $i \geq 1$, $H_i(\Lambda^*_B)$ may be computed by hand or using a suitable software such as Maple (Maple V Release 4, Waterloo Maple Inc., June 1996, Gröbner package). A set of generators of $H_1$ is, $z_1 = (-2^{n+c-r} - 2^{n-1-r}\delta_c)x_1 + (2^{c+1-r}b + 2^{-r}b\delta_c)x_2$, $r = min\{n-1, c+1\}$, $z_2 = (2+y)x_1$, and $z_3 = 2^{n-\alpha}x_1 + 2^{-\alpha}byx_2$.

From this and by calculating $H_i, i > 1$ from the Koszul resolution, it is not hard to verify that as graded algebras

$$Tor^*_{A"}(B; Z) \approx \Lambda^*_Z[z_1, z_2, z_3] \otimes Z[y, \delta_c]/I \tag{27}$$

with $z_1, z_2,\ z_3$ of degree 1, $y$ and $\delta_c$ of degree 0, and $I$ the ideal generated by $y^2 + 2y$; $z_2 z_3$; $yz^2$; $\delta_c^2 + 2^{c+1}\delta_c - 2^{2c-1} yL$; $2^\alpha y$; $2^{s-1}(2 + y)\delta_c + 2^{n-1} y$; $z_2 z_3$; $2^r \delta_c z_1 + 2^{2c-2+\alpha} Lyz_3$; $2^r yz_1 + 2^{n-1}\delta_c z_2 + 2^\alpha(2^c y - \delta_c)z_3$; $-2^{n-\alpha} z_2 + (2 + y)z_3$; $2^\alpha z_3 - (2^{n-1} + 2^{s-1}\delta_c)z_2$.

Hence we have

**Theorem 5.** *The Hodgkin spectral sequence for $X_{2n+1,2s}$ collapses, and so, as graded algebras,*

$$K^*(X_{2n+1,2s}) \approx \Lambda^*_Z[t_1, \cdots, t_{s-2}, z_1, z_2, z_3] \otimes Z[y, \delta_c]/I,$$

*where $I$ is the ideal generated by the elements listed in (27).*

*Remark* 4. The element $y + 1 \in K^*(X_{m,k})$ can be identified with the complexified Hopf bundle $c\xi_{m,k}$ over $X_{m,k}$ (see [2]). Then it follows from the preceding results that for $mk$ even the order of $c\xi_{m,k}$ is

$$2^{\alpha(m,k)} = g.c.d. \left\{ 2^{[(m-1)/2]},\ 2^{2i-1}\binom{n}{i},\ i = c + 1, \cdots, [(m - 3)/2] \right\}.$$

## 8. $K^*(X_{m,k})$ for $mk$ odd

When $mk$ is odd, various things go wrong. First, the Pontrjagin classes do not make sense, so we are obliged to work with the exterior powers $\bar{\lambda}_i = \bar{\Lambda}^i - \binom{2c}{i}$ instead (recalling that we are writting $\overline{\Lambda}[t]$ for the $\Lambda[t]$ corresponding to $RH_{m,k}[t]$ in order to distinguish the $\Lambda[t]$ in $RSpin(2n + 1)[t]$). Second, there is the more serious problem that $RH$ is no longer a free $Z[\lambda_1, \cdots, \lambda_c]$-module, so that change of rings cannot be used.

We shall therefore content ourselves with calculating $K^0(X_{m,k})$ and, in particular, the order of $y$ and $\delta_c$ in $K^*(X_{m,k})$. The relevant complex is $\Lambda^*_{RH}[x_1, \cdots, x_n]$, which is given by

$$0 \to \Lambda^n_{RH}(x_1, \cdots, x_n) \to \cdots \to \Lambda^1_{RH}(x_1, \cdots, x_n) \to \Lambda^0_{RH}(x_1, \cdots, x_n) = RH,$$

where $d(x_i) = Res(\lambda_i) = \lambda'_i$ $(0 < i < n)$ and $d(x_n) = Res(\delta_n) = 2^s\delta_c$. Then, since $H_0(\Lambda^*_{RH}) = RH/\operatorname{Im} d$, we have

$$H_0(\Lambda^*_{RH}) = Z[\lambda_1, \cdots, \lambda_{c-1}, \delta_c, y]/(\lambda'_1, \cdots, \lambda'_{n-1}, 2^s\delta_c, y^2 + 2y, y\delta_c + 2^c y).$$

In $RH$ we have, recalling Lemma 1, Cor. 1, and 4.3, Cor. 1, that $\bar{\Lambda}^j = \theta\bar{\Lambda}^{2c-j}$ for all $j$ and $\bar{\Lambda}^c = \bar{\Delta}_c^2 - (1 + \theta)(1 + \bar{\Lambda}^1 + ... + \bar{\Lambda}^{c-1})$ (Theorem 2). Also note that $y\bar{\Delta}_c = (\theta - 1)\bar{\Delta}_c = \bar{\Delta}_c - \bar{\Delta}_c = 0$.

Let $\lambda'[t] = \Lambda'[t] - (1 + t)^{2n+1} = \sum_{i>0} \lambda'_i t^i$ ($\lambda'_i = \chi_i \lambda'[t]$, where $\chi_i$ stands for "coefficient of $t^i$" as usual). Using this and applying (9), we have

$$\lambda'[t] = (1 + t\theta)^{2s+1}(\bar{\Lambda}[t] - f(t)),$$

where $f(t) = F(\theta, t)$, with (by definition)

$$F(z, t) = \frac{(1 + t)^{2n+1}}{(1 + zt)^{2s+1}}.$$

Thus

$$\chi_i(1+t\theta)^{-2s-1}\lambda'[t] = \chi_i(\bar{\Lambda}[t] - f(t)) = \overline{\Lambda}^i - \chi_i f(t), \quad i>0. \tag{28}$$

Now in $H_0(\Lambda^*_{RH})$ we have $\lambda'_i = 0$, $0<i<n$. Since $\chi_i(1+t\theta)^{-2s-1}\lambda'[t]$ is a $Z[\theta]$-linear combination of $\lambda'_1, \cdots, \lambda'_{n-1}$, from (28) we have

$$\overline{\Lambda}^i - \chi_i f(t) = 0, \qquad 0<i<n. \tag{29}$$

But in $RH$, $\overline{\Lambda}^{c+i} = \theta\overline{\Lambda}^{c-i}$, $i \ge 0$. Substituting this in (29), we have, in $H_0(\Lambda^*_{RH})$,

$$(\chi_{c+i} - \theta\chi_{c-i})f(t) = 0, \qquad i \ge 0. \tag{30}$$

In particular,

$$y\chi_c f(t) = 0. \tag{31}$$

Our next objective is to replace the relations (30) and (31) by the single relation $by$, where

$$b = g.c.d. \left\{ 2^{2c}\binom{n}{c},\ 2^{2i-1}\binom{n}{i},\ i = c+1, \cdots, n-1 \right\}.$$

For $i \ge 0$, $(\chi_{c+i} - z\chi_{c-i})F(z,t)$ is a polynomial, $h_i(z)$, say. Clearly, $h_i(1) = (\chi_{c+i} - \chi_{c-i})(1+t)^{2c} = \binom{2c}{c+i} - \binom{2c}{c-i} = 0$. Since $h_i(1) = h_i(-1)$ modulo 2 and $\theta^2 = 1$,

$$\begin{aligned} h_i(\theta) &= \frac{1}{2}(h_i(1) + h_i(-1)) + \frac{\theta}{2}(h_i(1) - h_i(-1)) \\ &= \frac{1+\theta}{2}h_i(1) + \frac{1-\theta}{2}h_i(-1) = 0 - \frac{1}{2}yh_i(-1). \end{aligned}$$

Now $h_i(-1) = (\chi_{c+i} + \chi_{c-i})F_{s,c}$ where $F_{s,c} = (1+t)^{2c}\left(\frac{1+t}{1-t}\right)^{2s+1}$. It follows that

$$\begin{aligned} y\chi_c f(t) &= (\theta - 1)\chi_c f(t) = -(1-\theta)\chi_c F(\theta, t) \\ &= -(\chi_c - \theta\chi_c)F(\theta,t) = -h_0(\theta) = \frac{1}{2}yh_0(-1) = y\chi_c F_{s,c}, \end{aligned}$$

and also

$$(\chi_{c+i} - \theta\chi_{c-i})f(t) = h_i(\theta) = -\frac{1}{2}yh_i(-1) = -\frac{1}{2}(\chi_{c+i} + \chi_{c-i})F_{s,c}.$$

**Lemma 7.** *With the above notation,*

*(i)* $\chi_c F_{s,c} = 2^{2c}\binom{s+c}{c}$, *all* $s \ge 0$, *all* $c \ge 0$.

*(ii) For all* $i > 0$ *we have* $(\chi_{c+i} + \chi_{c-i})F_{s,c} = \chi_{c+i}F_{s,c+i} - \binom{2i}{i}\chi_c F_{s,c} + Z$*-linear combination of* $(\chi_{c+j} + \chi_{c-j})F_{s,c}$ $(0<j<i)$.

*Proof.* (i) By induction on $n = s+c$ starting with $n=0$ and using

(a) $\chi_c F_{s,c} = \chi_c F_{s-1,c} + 4\chi_{c-1}F_{s,c-1}$ (this follows from $F_{s,c} - F_{s-1,c} = 4tF_{s,c-1}$),

(b) $\chi_0 F_{s,0} = 1$, all $s \ge 0$,

(c) $\chi_c F_{0,c} = \chi_c\left(\frac{(1+t)^{2c+1}}{1-t}\right) = 2^{2c}$, $\forall c \ge 0$ (this follows using $\frac{1}{1-t} = \sum_{i \ge 0} t^i$).

(ii) We have, for $i > 0$,

$$(1+t)^{2i} = 1 + t^{2i} + \binom{2i}{i}t^i + \sum_{0<j<i}\binom{2i}{j}t^j\left(1 + t^{2(i-j)}\right).$$

Hence $(\chi_{c+i} + \chi_{c-i})F_{s,c} = \chi_{c+i}(1+t^{2i})F_{s,c}$ and the result follows. ∎

From the lemma we deduce (respectively from (i), (ii))

1. $y\chi_c f = 2^{2c}\binom{s+c}{c}y$.

2. For $i>0$, $\frac{1}{2}h_i(-1) = 2^{2(c+i)-1}\binom{s+c+i}{i} - \frac{1}{2}\binom{2i}{i}2^{2c}\binom{s+c}{c} + Z$-linear combination of $\frac{1}{2}h_j(-1)$ $(0<j<i)$.

An easy inductive argument (using Pascal's triangle) now shows that

$$\begin{aligned} & g.c.d.\left\{2^{2c}\binom{s+c}{c}, \frac{1}{2}h_i(-1)\ (0<i<n-c)\right\} \\ = \quad & g.c.d.\left\{2^{2c}\binom{s+c}{c}, 2^{2(c+i)-1}\binom{s+c}{c+i}\ (0<i<n-c)\right\}. \end{aligned}$$

Writing $b$ for this, we conclude that the relations $y\chi_c f(t)$, $(\chi_{c+i}-\theta\chi_{c-i})f(t)$ $(0<i<n-c)$ may be replaced by the single relation $by$.

The next item on the agenda is the relation involving $\bar{\Delta}_c^2$ . In $H_0(\Lambda^*_{RH})$ we have

$$\begin{aligned} \bar{\Delta}_c^2 &= (1+\theta)(1+\bar{\Lambda}^1+\cdots+\bar{\Lambda}^{c-1}) + \overline{\Lambda}^c \qquad (32) \\ &= (1+\theta)(1+\chi_1+\cdots+\chi_{c-1})f(t) + \chi_c f(t). \end{aligned}$$

But

$$\begin{aligned} (1+\theta)\chi_i f(t) = \chi_i(1+\theta)f(t) &= \chi_i(1+\theta)(1+t)^{2c}\left(\frac{1+t}{1+\theta t}\right)^{2s+1} \\ &= \chi_i(1+\theta)(1+t)^{2c} \end{aligned}$$

since $\theta(1+\theta)=(1+\theta)$ and using the binomial theorem. Substituting this in the relation (32) for $\bar{\Delta}_c^2$, we have

$$\begin{aligned} \bar{\Delta}_c^2 &= (1+\theta)(1+\chi_1+\cdots+\chi_{c-1})(1+t)^{2c} + \chi_c f(t) \qquad (33) \\ &= (1+\theta)\sum_{i=0}^{c-1}\binom{2c}{c} + \chi_c f(t) = (2+y)\frac{1}{2}\left[2^{2c}-\binom{2c}{c}\right] + \chi_c f(t). \end{aligned}$$

Now

$$\begin{aligned} \chi_c f = \chi_c F(\theta,t) &= \chi_c\left(\frac{F(1,t)+F(-1,t)}{2}\right) + \theta\chi_c\left(\frac{F(1,t)-F(-1,t)}{2}\right) \\ &= \frac{1}{2}\left(\binom{2c}{c} + 2^{2c}\binom{n}{c}\right) + \frac{1}{2}\left(\binom{2c}{c} - 2^{2c}\binom{n}{c}\right)\theta \end{aligned}$$

(using Lemma 7 for $F(-1,t)$). Hence from (33) in $H_0(\Lambda^*_{RH})$ we have

$$\bar{\Delta}_c^2 = 2^{2c-1}y\left[1-\binom{n}{c}\right] + 2^{2c},$$

or, equivalently,

$$\delta_c^2 + 2^{c+1}\delta_c = 2^{2c-1}y\left[1-\binom{n}{c}\right].$$

Thus,

$$K^0(X_{m,k}) = Z[\bar{\Delta}_c, y]/(y^2+2y,\ \bar{\Delta}_c y,\ 2^s\delta_c,\ by,\ \bar{\Delta}_c^2 - D),$$

where $D = 2^{2c} + 2^{2c-1}\left\{1-\binom{n}{c}\right\}y$.

Let $R$ be the ring $Z[y]/(y^2+2y, by)$. Then

$$K^0(X_{m,k}) = R[\bar{\Delta}_c]/(y\bar{\Delta}_c,\ 2^s\delta_c,\ \bar{\Delta}_c^2 - D).$$

An easy calculation shows that $R = Z \oplus Z_b y$ ($Z_b$ being $Z/b$).

Consider the inclusion $R + R\bar{\Delta}_c$ in $R[\bar{\Delta}_c]$. This induces an epimorphism $R + R\bar{\Delta}_c \to K^0(X_{m,k})$ (because of the relations $y\bar{\Delta}_c$, $\bar{\Delta}_c^2 - D$). Now $r_0 + r_1\bar{\Delta}_c$ lies in the kernel if and only if $r_0 + r_1\bar{\Delta}_c = Fy\bar{\Delta}_c + G(2^s\bar{\Delta}_c - 2^n) + H(\bar{\Delta}_c^2 - D)$, where $r_0, r_1 \in R$ and $F, G, H \in R[\bar{\Delta}_c]$. Writing $F = F_0(\bar{\Delta}_c^2) + \bar{\Delta}_c F_1(\bar{\Delta}_c^2)$ and similarly for $G$ (note that any $F, G \in R[\bar{\Delta}_c]$ can be written this way), we see that the expression

$$\left(F_0(\bar{\Delta}_c^2) + \bar{\Delta}_c F_1(\bar{\Delta}_c^2)\right)\bar{\Delta}_c y + \left(G_0(\bar{\Delta}_c^2) + \bar{\Delta}_c G_1(\bar{\Delta}_c^2)\right)(2^s\bar{\Delta}_c - 2^n)$$
$$-\{F_0(D)\bar{\Delta}_c y + DF_1(D)y + 2^s G_0(D)\bar{\Delta}_c - 2^n\bar{\Delta}_c G_1(D) - 2^n G_0(D) + 2^s D G_1(D)\}$$

is divisible by $\bar{\Delta}_c^2 - D$. Thus, for a suitable choice of $H$,

$$r_0 + r_1\bar{\Delta}_c = F_0(D)\bar{\Delta}_c y + G_0(D)\left\{2^s\bar{\Delta}_c - 2^n\right\} + G_1(D)\left\{2^s D - 2^n\bar{\Delta}_c\right\}$$

(here we used the fact that $Dy = 0$ in $R$, which follows easily from $by = 0$).

Now $F_0(D)$, $G_0(D)$, $G_1(D)$ are arbitrary elements of $R$. Hence the kernel above is generated over $Z$ by $\bar{\Delta}_c y$, $\bar{\Delta}_c y^2$, $2^s\bar{\Delta}_c - 2^n$, $2^s\bar{\Delta}_c y - 2^n y$, $2^s D - 2^n\bar{\Delta}_c$ and $2^s Dy - 2^n\bar{\Delta}_c y$, or, equivalently, by $\Delta_c y$, $2^s\delta_c$, $2^n y$, $2^s D - 2^n\bar{\Delta}_c$, that is, by $\bar{\Delta}_c y$, $2^s\delta_c$, $2^n y$, $2^{s+2c-1}\left\{1 - \binom{n}{c}\right\} y = 2^{n+c-1}[1 - \binom{n}{c}]y$. But the last term is zero even if $s = 0$ (for then $\binom{n}{c} = 1$). Thus,

$$\begin{aligned} K^0(X_{m,k}) &= (R + R\bar{\Delta}_c)/(2^s\delta_c, \Delta_c y, 2^n y) = (R + Z\bar{\Delta}_c)/(2^s\delta_c, 2^n y) \\ &= (Z + Z_b y + Z\delta_c)/(2^s\delta_c, 2^n y) = Z + Z_{2^\alpha} y + Z_{2^s}\delta_c, \end{aligned}$$

where

$$2^\alpha = g.c.d.\left\{2^n;\ 2^{2c}\binom{n}{c};\ 2^{2(c+i)-1}\binom{n}{c+i},\ 0 < i < n - c\right\}.$$

Furthermore, $y^2 = -2y$, $y\delta_c = -2^c y$, and $\delta_c^2 = -2^{c+1}\delta_c + 2^{2c-1}[1 - \binom{n}{c}]y$.

We end by observing that:

1. The generators occurring in Theorems 4 and 5 may be described geometrically via the $\alpha$ and $\beta$ constructions, as in [10] and [12].

2. Using the Gysin sequence, $K^*(X_{2n,2s})$ may be deduced from $K^*(X_{2n,2s-1})$ as in [7], but $K^*(X_{2n+1,2s+1})$ cannot be similarly deduced, as $X_{2n+1,2s+1}$ is non-orientable.

3. In the special cases $X_{2n,2n-1} = PO(2n)$ and $X_{m,1} = RP^{m-1}$ these results reduce to those of [9], [10] and [1]. For $X_{4n,2k-1}$ they agree with [2].

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